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Travelling with/against the Flow. Deterministic Diffusive Driven Systems

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Abstract We introduce and study a deterministic lattice model describing the motion of an infinite system of oppositely charged particles under the action of a constant electric field. As an application this model represents a traffic flow of cars moving in opposite directions along a narrow road. Our main results concern the Fundamental diagram of the system describing the dependence of average particle velocities on their densities and the Phase diagram describing the partition of the space of particle configurations into regions having different qualitative properties, which we identify with free, jammed and hysteresis phases.

Keywords Diffusive driven system · Traffic flow · Phase transition

1 Introduction

It is well known that far from equilibrium statistical physics systems may demonstrate a very complex behavior even on the level of their stationary states. In general, such non-equilibrium stationary (steady) states depend sensitively on the details of the microscopic dynamics which leads to a very diverse organization at the macroscopic level. An important class of such systems is represented by *driven diffusive systems* (DDS), describing the behavior of interacting diffusing particles driven into selected spatial directions by an external force. Models of this sort are widely used to describe vehicular and pedestrian traffic [7, 8, 14, 21], water droplets in microemulsions with distinct charges [2], and numerous biological problems from molecular motors to protein synthesis (see e.g. [1, 23]).

Basic ideas used in these models were introduced originally in [19] as a simple modification of the Ising lattice gas. From that time tens of publications have been devoted to

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this subject. From the point of view of mathematics these processes are close relatives to asymmetric exclusion processes and topological Markov chains.

The simplest model of this sort can be described as follows. Consider a finite one-dimensional lattice with periodic boundary conditions (i.e. a ring). Each site of the lattice is either vacant ($\cdot$) or occupied by a positive ($+$) or negative ($-$) particle. Positive particles are driven to the right and negative particles are driven to the left. The dynamics is random-sequential, i.e. at each time step a pair of nearest neighbor sites is selected at random and particles or vacancies located at these sites are exchanged according to the following set of probabilities:

$$\mathcal{P}(+- \ \to \ +-) = \beta, \qquad \mathcal{P}(-+ \ \to \ +-) = 0,$$
$$\mathcal{P}(+\cdot \ \to \ \cdot+) = \mathcal{P}(\ \cdot- \ \to \ -\cdot) = 1.$$

Here the parameter $0 < \beta \le 1$ plays the role the time delay required to exchange neighboring positive and negative particles $+-$. Numerical studies and mean-field type approximations of this model and its two-lane generalizations have been a subject of a large number of publications (see e.g. [1, 2, 9–13, 15–20] and further references therein).

Our aim is to introduce a deterministic version of a DDS and to study rigorously its limit dynamics (as time goes to infinity) on the infinite integer lattice $\mathbb{Z}$. In order to do this we first pass from the random-sequential to synchronous movement of particles (when all particles are trying to move simultaneously), and then construct deterministic exchange rules for particles of different signs. The simplest way to achieve the second goal is to replace the random exchange with the introduction of the waiting time $\tau = 1/\beta$ before the exchange. (Indeed, the average waiting time in the random exchange version is exactly $1/\beta$.) Further on we assume that the parameter τ takes only positive integer values.

To define the synchronous movement of particles one needs to resolve the problem with the presence of triples $+\ \cdot -$ (positive and negative particles separated by a single vacancy) additionally to couples $+-$ considered in the asynchronous case. In the case of a triple $+\ \cdot-$ both positive and negative particles are trying simultaneously to exchange their positions with the same vacancy (which cannot happen under the asynchronous updating). We assume that in this case the waiting time before the particles exchange their positions is equal to $\tau+1$.[1] The situations $+-$ and $+\ \cdot-$ we call *short* and *long interactions*, while the situations $++$ and $--$ when a particle is blocked by another particle of the same sign we call *simple interactions*. Using the notion of interactions the dynamics can be described as follows. All particles in a configuration are trying to move in the direction corresponding to its sign by one position. If the particle does not interact with others then it makes a move. If the short/long interaction takes place then the particle waits for τ or $\tau+1$ time steps (depending on the interaction type) before to start moving, while in the case of the simple interaction the particle simply preserves its position. As an example consider the dynamics of a spatially periodic configuration with $\tau = 2$ (only the main period of length 17 containing 4 positive ($+$) and 2 negative ($-$) particles and 11 vacancies (.) is shown):

```
+ . . + . . . . - . . + + . . . -      t = 0
. + . . + . . - . . . + . + . - .      t = 1
. . + . . + - . . . . . + + . - .      t = 2
. . . + . + - . . . . . + + . - .      t = 3
```

[1] One might argue that after the first time step both particles move to the vacant site and then use the standard waiting time τ to make further exchanges.

```
. . . . + − + . . . . . + − . + .     t = 4
. . . . + − . + . . . . + − . . +     t = 5
+ . . . − + . . + . . . − + . . .     t = 6
. + . − . . + . . + . − . . + . .     t = 7
. + . − . . . + . + . − . . . + .     t = 8
. + . − . . . . + + . − . . . . +     t = 9
+ − . + . . . . + − . + . . . . .     t = 10
+ − . . + . . . + − . . + . . . .     t = 11
− + . . . + . . − + . . . + . . .     t = 12
. . + . . . + − . . + . . . + . −     t = 13
. . . + . . + − . . . + . . + . −     t = 14
. . . . + . − + . . . . + . + . −     t = 15
```

Here the first two positive particles and the first negative particle move freely until the short interactions with the 1st negative particle consecutively take place at time $t = 2$ and $t = 4$. Then after 2 time steps the positive and negative particles exchange their positions. The 3rd positive particle at the 1st time step makes a simple interaction with the 4th positive particle. The 4th positive particle starting from $t = 1$ is making a long interaction with the 2nd negative particle during three time steps, then exchange their positions, etc. Note that due to the spatial periodicity after the 5th time step a positive particle "moves" from the 17th to the 1st position, while after the 12th time step a negative particle "moves" from the 1st to the 17th position.

In fact, this model is not Markov since the information about present positions of particles alone does not allow to predict the future motion of particles (due to the time delays during particle interactions) and to make the model precise one needs to introduce the notion of particles *states* which will be done in Sect. 2.

The analogy between the classical gas-liquid transition and stationary states in interacting particle systems is well known and discussed at length in the literature. The gaseous or laminar phase corresponds to the situation when particles almost never interact with each other and thus move with constant velocities. Typically this happens at low particle densities. In the liquid phase (which occurs normally at high densities) the particles are located close to each other and mutual interactions organize them into clusters (traffic jams). Additionally there might be an intermediate or hysteresis phase when both types of particle behavior coexist.

In our case the classical laminar phase cannot take place since particles moving in opposite directions interact inevitably. To overcome this difficulty we say that a particle is becoming *free* starting from a certain time $t_0 \geq 0$ if for $\forall t \geq t_0$ it does not take part in simple interactions. In other words, after an initial transient period of length t_0 the particle stops interacting with particles of the same sign. Then instead of the laminar phase we consider a pair of *eventually free* positive and negative phases defined as subsets $\mathrm{Free}_{\pm}^{\infty}$ of the set of all particle configurations as follows. A configuration belongs to $\mathrm{Free}_{+}^{\infty}$ if each its positive particle becomes free after a finite transient period, which might depend on the particle. The set $\mathrm{Free}_{-}^{\infty}$ is defined in the same way but for negative particles. Similarly instead of the liquid phase we consider two *eventually jammed* (notation $\mathrm{Jam}_{\pm}^{\infty}$) phases having the property that clusters of particles of the corresponding sign are always present after some transient period. The cases when the length of the transient period is equal to zero we denote by $\mathrm{Free}_{\pm}$ and $\mathrm{Jam}_{\pm}$ respectively. Note that the sets of eventually free and eventually jammed configurations of different signs have nonempty intersections and that it is possible that an initial configuration has no interacting particles but under dynamics they will start interacting and form "jams".

Our main results are formulated in the following two theorems. We start with qualitative results justifying our discussion of phase transitions and giving an inner characterization of

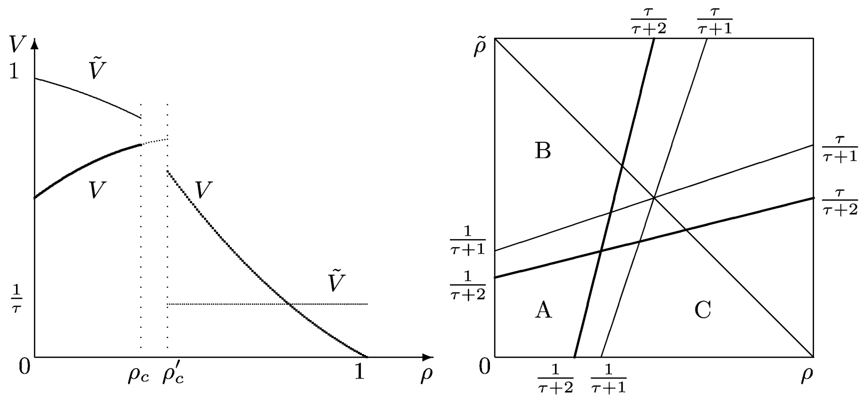

Fig. 1 (*Left*) Fundamental diagram: dependence of the limit average velocity V on the density ρ of positive particles with the fixed density $\tilde\rho < 1/(\tau+2)$ of negative particles. The average velocity $\tilde V$ of negative particles is indicated by a *thin line*. Values ρ_c, ρ_c' indicate boundaries of the hysteresis phase. (*Right*) Phase diagram: $A \subset \mathrm{Free}_+^\infty \cap \mathrm{Free}_-^\infty$, $B \subset \mathrm{Free}_+^\infty$, $C \subset \mathrm{Free}_-^\infty$. The region $H := \{0 \le \rho + \tilde\rho \le 1\} \setminus (A \cup B \cup C)$ between thick and thin lines belongs to the hysteresis phase

individual configurations belonging to each phase. To this end in Sect. 3 we introduce the notion of signed (positive and negative) (proto)clusters[2] of particles. By the life-time of a (proto)cluster one means the duration of time until it ceases to exist. Exact definitions and discussion see in Sects. 2, 3.

Theorem 1.1 *Let for a configuration x the densities of positive and negative particles $(\rho(x), \tilde\rho(x))$ be well defined. If*

(a) $(\tau+2)\rho(x) < 1 + (\tau-1)\tilde\rho(x)$ *then* $x \in \mathrm{Free}_+^\infty$.
(b) $(\tau+2)\tilde\rho(x) < 1 + (\tau-1)\rho(x)$ *then* $x \in \mathrm{Free}_-^\infty$.
(c) $(\tau+1)\rho(x) > 1 + (\tau-1)\tilde\rho(x)$ *then* $x \in \mathrm{Jam}_+^\infty$.
(d) $(\tau+1)\tilde\rho(x) > 1 + (\tau-1)\rho(x)$ *then* $x \in \mathrm{Jam}_-^\infty$.

To formulate quantitative results consider a partition of the triangle $\Delta := \{(\rho, \tilde\rho) :\ 0 \le \rho + \tilde\rho \le 1,\ \rho, \tilde\rho \ge 0\}$ (describing all possible pairs of densities) made by 4 straight lines:

$$(\tau+1)\rho = 1 + (\tau-1)\tilde\rho, \qquad (\tau+2)\rho = 1 + (\tau-1)\tilde\rho,$$
$$(\tau+1)\tilde\rho = 1 + (\tau-1)\rho, \qquad (\tau+2)\tilde\rho = 1 + (\tau-1)\rho$$

(see Fig. 1 (right)). Denote

$$A := \{(\rho, \tilde\rho) \subset \Delta : (\tau+2)\rho < 1 + (\tau-1)\tilde\rho, (\tau+2)\tilde\rho < 1 + (\tau-1)\rho\},$$
$$B := \{(\rho, \tilde\rho) \subset \Delta : (\tau+2)\rho < 1 + (\tau-1)\tilde\rho, (\tau+1)\tilde\rho > 1 + (\tau-1)\rho\},$$
$$C := \{(\rho, \tilde\rho) \subset \Delta : (\tau+1)\rho > 1 + (\tau-1)\tilde\rho, (\tau+2)\tilde\rho < 1 + (\tau-1)\rho\},$$
$$H := \Delta \setminus (A \cup B \cup C).$$

[2]Roughly speaking a protocluster is a collection of particles of the same sign which will form a true cluster in future. A configuration $x \in \mathrm{Jam}_\pm^\infty$ may contain no infinite life-time clusters but in that case there are infinite life-time protoclusters.

Theorem 1.1 implies that $A \subset \mathrm{Free}_+^\infty \cap \mathrm{Free}_-^\infty$, $B \subset \mathrm{Free}_+^\infty \cap \mathrm{Jam}_-^\infty$, $C \subset \mathrm{Free}_-^\infty \cap \mathrm{Jam}_+^\infty$.

Theorem 1.2 *Let the densities of both positive and negative particles $\rho, \tilde\rho$ in the initial configuration be well defined. Then the corresponding average particle velocities $V, \tilde V$ are well defined as well and depend only on the particle densities according to the following relations*:

$$V(\rho,\tilde\rho) = \frac{1+(\tau-1)(\rho-\tilde\rho)}{1+(\tau-1)(\rho+\tilde\rho)}, \quad \tilde V(\rho,\tilde\rho) = \frac{1+(\tau-1)(\tilde\rho-\rho)}{1+(\tau-1)(\rho+\tilde\rho)} \quad \textit{if } (\rho,\tilde\rho)\in A,$$

$$V(\rho,\tilde\rho) = \frac{1}{\tau}, \quad \tilde V(\rho,\tilde\rho) = \frac{1}{\tau}\left(\frac{1}{\tilde\rho}-1\right) \quad \textit{if } (\rho,\tilde\rho)\in B,$$

$$V(\rho,\tilde\rho) = \frac{1}{\tau}\left(\frac{1}{\rho}-1\right), \quad \tilde V(\rho,\tilde\rho) = \frac{1}{\tau} \quad \textit{if } (\rho,\tilde\rho)\in C.$$

Note that Theorem 1.1 gives a qualitative description of the asymptotic dynamics, while Theorem 1.2 describes it in quantitative terms. It is worth mention also that without a systematic preliminary numerical modelling which allow us to understand qualitatively the local structure of the particle flow even the formulation of results proven in this paper would be impossible.

Proofs of these results ideologically are based on the machinery of the analysis of life-times of particle clusters developed in [4–6] where deterministic interacting particle systems with a single type of particles were studied. The presence of particles moving in the opposite direction together with two types of interactions (short and long) complicates significantly the behavior of the system. In particular, new clusters may be born in a free flow of particles, and the asymptotic dynamics depends on two parameters (densities of positive and negative particles). Therefore the technics has been changed a lot and yet we are able to give only lower and upper estimates of the life-times. Nevertheless these estimates allow to find exact boundaries of all phases present in the model.

The paper is organized as follows. In Sect. 2 we give the formal description of the model and main statistical quantities under study: particle densities, average velocities, etc. In the absence of clusters the dynamics of particles is trivial (except from interactions between particles of opposite signs). Therefore to study the model we need to analyze the dynamics of clusters of particles and their "life-times". Exact definitions of these objects and corresponding mathematical results are discussed in Sect. 3. Duality relations between positive and negative particles allow us to calculate in Sect. 4 average particle velocities under the assumption that they are well defined. The latter is connected to the proof of our main results formulated in the Introduction and given in Sect. 5. Section 6 is dedicated to the analysis of the region of the Phase diagram where different phases coexist. In Sect. 7 we apply our results to study a model of an active tracer moving with or against the particle flow, and Sect. 8 is dedicated to the generalization of qualitative results for the case of configurations for which particle densities are not well defined.

2 The Model

The main disadvantage of the model discussed in the Introduction is that the information about the present positions of particles alone without the knowledge for how long currently occurring interactions already take place does not allow to define the future motion of particles. In order to overcome this difficulty we introduce the notion of a state of a particle which takes into account the complete information about the occurring interactions. Let us give formal definitions.

By a *configuration* we mean a bi-infinite sequence $x = \{x_i\}_{-\infty}^{\infty}$ with elements from the alphabet $\{-\tau-1, -\tau, \dots, -1, 0, 1, \dots, \tau, \tau+1\}$. Positive entries correspond to positive particles, negative entries to negative ones, while zero entries correspond to vacancies. Non zero entries will be referred as *states* of particles located at corresponding sites. The states will be used to take into account the delays during interactions of positive and negative particles. Thus the largest value of the state $\tau+1$ corresponds to the long interaction.

The set of *admissible* configurations X consists only of configurations x satisfying the condition that for each $i \in \mathbb{Z}$

- if $x_i > 1$ then either $x_{i+1} = -x_i$ or $x_{i+1} = 0$ and $x_{i+2} = -x_i$, additionally $x_i = \tau + 1$ implies $x_{i+1} = 0$;
- if $x_i < -1$ then either $x_{i-1} = -x_i$ or $x_{i-1} = 0$ and $x_{i-2} = -x_i$, additionally $x_i = -\tau - 1$ implies $x_{i-1} = 0$.

These conditions imply restrictions only to positions of interacting particles having states greater than one on modulus. For example, the configuration $\dots 110\tilde{1}01\tilde{1}00020\tilde{2}00 \dots$ is admissible (here $\tilde{1}$ and $\tilde{2}$ stand for -1 and -2), while $\dots 110\tilde{1}01\tilde{1}00020\tilde{1}00 \dots$ is not, since the last two particles separated by a single vacancy $20\tilde{1}$ are supposed to be mutually interacting (according to their positions) but their states differ on modulus.

By $x^t := T^t x,\ t \in \mathbb{Z}_+ \cup \{0\}$ we denote the state of the configuration x at time t, assuming that the initial state x^0 is given by x. Here $T : X \to X$ is the map describing the dynamics which we define on the level of individual particles in the configuration $x \in X$ in the following three steps:

1. First consider sites i, i' with $|i' - i| \le 2$ containing mutually interacting particles with $x_i^t > 1$ and $x_{i'}^t = -x_i^t$ and set $x_i^{t+1} := x_i^t + 1,\ \ x_{i'}^{t+1} := x_{i'}^t - 1$. Then if $i' - i = 1$ (short interaction) and $x_i^{t+1} > \tau$ or if $i' - i = 2$ (long interaction) and $x_i^{t+1} > \tau + 1$ set $x_i^{t+1} := -1,\ \ x_{i'}^{t+1} := 1$.
2. Then consider the sites i with $x_i^t = 1$.
   (a) if $x_{i+1}^t = 0$ and $x_{i+2}^t \ne -1$ then set $x_i^{t+1} := 0,\ x_{i+1}^{t+1} := 1$;
   (b) if $x_{i+1}^t = 0$ and $x_{i+2}^t = -1$ then set $x_i^{t+1} := 2,\ x_{i+2}^{t+1} := -2$;
   (c) otherwise if $x_{i+1}^t = -1$ then set $x_i^{t+1} := 2,\ x_{i+1}^{t+1} := -2$.
3. It remains to consider the sites i with $x_i^t = -1$ which were not taken into account during the step 2. If $x_{i-1}^t = 1$ set $x_{i-1}^{t+1} := -1,\ x_i^{t+1} := 0$; otherwise do nothing.

In words, if a particle is not interacting it simply moves by one position in the direction corresponding to its sign (rule 2.a and 3). In case of the simple interaction (the particle is blocked by another particle of the same sign) it does not move an does not change its state. If the short/long interaction takes place the particle preserves its position but its state changes by ± 1 depending on the particle sign (see rule 1) until it reaches on modulus the value τ (in the case of the short interaction) or $\tau + 1$ (in the case of the long interaction). After that the particles get the states ± 1 (preserving original signs) and exchange their positions. The rules 2.b and 2.c take care about the initial stage of interactions.

Using the notation $\tilde{1} = -1, \tilde{2} = -2, \tilde{3} = -3$ we can rewrite the example of dynamics of a spatially periodic configuration (see the formal definition below) with $\tau = 2$ described in the Introduction as follows:

$$
\begin{array}{ll}
10010000\tilde{1}00110001\tilde{} & t = 0 \\
0100100\tilde{1}000101\tilde{1}0 & t = 1
\end{array}
$$

$$\begin{array}{ll}
001001\tilde{1}00000120\tilde{2}0 & t=2 \\
000102\tilde{2}00000130\tilde{3}0 & t=3 \\
000011\tilde{1}1000001\tilde{1}010 & t=4 \\
000022\tilde{2}0100002\tilde{2}001 & t=5 \\
1000\tilde{1}1001000\tilde{1}1000 & t=6 \\
010\tilde{1}0010010\tilde{1}00\tilde{1}00 & t=7 \\
020\tilde{2}0001020\tilde{2}00010 & t=8 \\
030\tilde{3}0000130\tilde{3}00001 & t=9 \\
1\tilde{1}0100001\tilde{1}0100000 & t=10 \\
2\tilde{2}0010002\tilde{2}0010000 & t=11 \\
\tilde{1}1000100\tilde{1}10001000 & t=12 \\
0010001\tilde{1}00100010\tilde{1} & t=13 \\
0001002\tilde{2}00010020\tilde{2} & t=14 \\
000010\tilde{1}100001030\tilde{3} & t=15
\end{array}$$

Invariance of the set of admissible configurations follows immediately from the definition of the map T. Observe also that the restriction of the map T to the subset of admissible configurations containing only entries from the alphabet $\{0, 1\}$ coincides with the classical Nagel-Schreckenberg traffic flow model (see e.g. [5, 22]).

For a configuration $x \in X$ by the *density* of positive particles $\rho(x, I)$ in a finite lattice segment[3] $I = [n, m] := \{i \in \mathbb{Z} : \ n \le i \le m\}$ we mean the number of positive particles from the configuration x located in I divided by the total number of sites in I (notation $|I|$), and by $\tilde{\rho}(x, I)$ the corresponding value for the negative particles. If for any sequence of *nested* finite lattice segments $\{I_n\}$ with $|I_n| \stackrel{n\to\infty}{\longrightarrow} \infty$ the limits

$$\rho(x) := \lim_{n\to\infty} \rho(x, I_n), \qquad \tilde{\rho}(x) := \lim_{n\to\infty} \tilde{\rho}(x, I_n)$$

are well defined and do not depend on $\{I_n\}$ we call $\rho(x)$ the *density* of positive particles and $\tilde{\rho}(x)$ the density of negative particles in the configuration $x \in X$. Otherwise one considers upper and lower particle densities $\rho^{\pm}(x)$, $\tilde{\rho}^{\pm}(x)$.

Lemma 2.1 *Particle densities are conserved under dynamics, i.e.* $\rho^{\pm}(x^t), \tilde{\rho}^{\pm}(x^t)$ *do not depend on* $t < \infty$.

Proof For a given lattice segment $I \in \mathbb{Z}$ the number of particles from the configuration $x^t \in X$ which can leave it during the next time step cannot exceed 2 and the number of particles which can enter this segment also cannot exceed 2. A close look shows that the total number of particles that can leave or enter this segment cannot exceed 2, because if a particle leaves the segment through one of its ends no other particle can enter through the same end. Therefore

$$|\rho(x^t, I) - \rho(x^{t+1}, I)| \cdot |I| \le 2$$

which implies the claim in the case of positive particles. The proof in the case of negative particles is exactly the same. □

A configuration $x \in X$ is said to be *spatially periodic* with *period* $L \in \mathbb{Z}$ if $x_i = x_{i+L}$ $\forall i \in \mathbb{Z}$. Such configurations represent an important class of admissible configurations for

[3] We shall use also the notation $x[n, m]$ for lattice segments to specify the configuration x.

which both positive and negative particle densities are well defined. A direct check demonstrates that the spatial periodicity and its period[4] are preserved under dynamics. Therefore the dynamics of spatially periodic configurations of period L is equivalent to the dynamics on the finite segment of the integer lattice of length L with periodic boundary conditions (i.e. to the only situation suitable for numerical modelling). The total number of admissible configurations on this lattice segment cannot exceed $L^{2\tau+3}$, which implies eventual periodicity in time[5] of the dynamics. Thus a trajectory $\{T^t x\}_{t\geq 0}$ of the original infinite system with a spatially periodic initial configuration $x \in X$ is eventually periodic in time as well.

We say that particles of the same sign located at sites $i' < i$ are *consecutive* if all sites between them are either vacant or occupied by particles of the opposite sign. Consider a pair of consecutive positive particles. The following result shows that the distance between these particles (calculated as $i - i' - 1$) can shrink to zero with time but cannot be enlarged much. (Compare with the incompressibility property of a conventional fluid!)

Lemma 2.2 *Let $\hat{X}$ be a collection of admissible configurations x having a pair of consecutive positive particles at sites $i' < i$. Denote by $i'(x^t), i(x^t)$ positions of these particles at time t in the configuration x^t and by $D(x^t) := i(x^t) - i'(x^t) - 1$ the distance between them. Then*

$$0 = \inf_{x\in\hat{X}} \liminf_{t\to\infty} D(x^t) < \sup_{x\in\hat{X}} \limsup_{t\to\infty} D(x^t) < 2\tau(i - i'). \tag{2.1}$$

Proof The lower estimate follows from the observation that, independently on the initial distance between the particles, there might be a large enough "jam" of positive particles ahead of them which will stop the leading particle and allow the rear one to catch up with it.

To prove the upper estimate observe that without interactions with negative particles the distance $D(x^t)$ cannot grow in time. Note also that a finite number of negative particles located initially in the segment $[i', i]$ can give only a constant contribution to the variation of the distance between the particles. Therefore it is enough to consider only the case when initially there are only vacancies in the segment $[i' + 1, i - 1]$. Denote by t_1 the first moment of time when the leading particle meets a negative one, by t_1' the duration of the interaction between them (which might take values τ or $\tau + 1$), by t_2 the time between the end of this interaction and the moment when the rear particle meets with the same negative particle, and by t_2' – the duration of the latter interaction. Then we have

$$\begin{aligned}
i'(x^{t_1+t_1'}) &= \max\{i'(x^{t_1}) + t_1', i(x^{t_1}) - 1\},\\
i(x^{t_1+t_1'}) &= i(x^{t_1}) + 1 + (t_1' - \tau),\\
i'(x^{t_1+t_1'+t_2}) &= i'(x^{t_1+t_1'}) + t_2,\\
i(x^{t_1+t_1'+t_2}) &\leq i(x^{t_1}) + 1 + (t_1' - \tau) + t_2,\\
i'(x^{t_1+t_1'+t_2+t_2'}) &= i'(x^{t_1+t_1'+t_2}) + 1 + (t_2' - \tau),\\
i(x^{t_1+t_1'+t_2+t_2'}) &\leq i(x^{t_1}) + 1 + (t_1' - \tau) + t_2 + t_2'.
\end{aligned}$$

[4]The *minimal* spatial period may decrease under dynamics.

[5]A trajectory becomes periodic in time after some finite transient period.

Therefore

$$
\begin{aligned}
D(x^{t_1+t_1'}) &= i(x^{t_1+t_1'}) - i'(x^{t_1+t_1'}) - 1 \\
&= \max\{i(x^{t_1}) - i'(x^{t_1}) - \tau + 1 - 1, 2\} \le \max\{D(x^{t_1}), 2\}, \\
D(x^{t_1+t_1'+t_2}) &= i(x^{t_1+t_1'+t_2}) - i'(x^{t_1+t_1'+t_2}) - 1 \\
&\le i(x^{t_1}) + (t_1' - \tau) + t_2 - \max\{i'(x^{t_1}) + t_1', i(x^{t_1}) - 1\} - t_2 \\
&\le \max\{i(x^{t_1}) - i'(x^{t_1}) - \tau - 2, 2\} \le \max\{D(x^{t_1}), 2\}, \\
D(x^{t_1+t_1'+t_2+t_2'}) &= i(x^{t_1+t_1'+t_2+t_2'}) - i'(x^{t_1+t_1'+t_2+t_2'}) - 1 \\
&\le i(x^{t_1}) + (t_1' - \tau) + t_2 + t_2' - \max\{i'(x^{t_1}) + t_1', i(x^{t_1}) - 1\} \\
&\quad - t_2 - 1 - (t_2' - \tau) \\
&= i(x^{t_1}) + t_1' - 1 - \max\{i'(x^{t_1}) + t_1', i(x^{t_1}) - 2\} \\
&\le \max\{i(x^{t_1}) - i'(x^{t_1}) - 1, t_1'\} \le \max\{D(x^{t_1}), \tau + 1\}.
\end{aligned}
$$

This implies the result since the contribution from a single negative particle located initially in the segment $[i', i]$ cannot exceed $\tau + 1$ and the number of such particles cannot be larger than $i - i' - 1$. □

Remark 2.3 Result of Lemma 2.2 is not obvious and is not what one expects here. Indeed, due to the presence of two types of interactions: short and long, one would expect that if the leading particle in the pair takes part only in short interactions while another one takes part only in long interactions, then the distance between them grows linearly in time. Lemma 2.2 demonstrates that this is not the case. An important property that we use here is that the difference between the duration of long and short interactions is exactly equal to one. If this would not be the case the distance between the particles indeed might grow with time.

Let us introduce now the notion of the *average velocity* of a particle. For a configuration x denote by i^t the location of a certain particle at time $t \ge 0$ located originally at site $i = i^0$. By a *finite time velocity* of this particle we mean $V(x, i, t) := \frac{1}{t}\,|i^t - i^0|$. If the limit $V(x, i) := \lim_{t\to\infty} V(x, i, t)$ is well defined we call it the *average velocity* of the particle. To distinguish between the positive and negative particles in the latter case we use the notation $\tilde{V}(x, i)$.

Our aim now is to show that the average velocity does not depend on the choice of a particle.

Lemma 2.4 *Let x be an admissible configuration and assume that for a positive particle originally located at site i the average velocity $V(x, i)$ is well defined. Then for any positive particle in the configuration x the average velocity is well defined and coincides with $V(x, i)$.*

Proof Denote by $\hat{i}$ the location at time $t = 0$ of the next positive particle located to the right from i. As usual we denote by $i^t, \hat{i}^t,\ t \ge 0$ positions of these particles at time $t \ge 0$. Then we have

$$
V(x, \hat{i}, t) = \frac{1}{t}\,(\hat{i}^t - \hat{i}^0) = \frac{1}{t}\,(i^t - i^0) + \frac{1}{t}\,(\hat{i}^t - i^t) + \frac{1}{t}\,(i^0 - \hat{i}^0).
$$

Applying Lemma 2.2, according to which $1 \le \hat{i}^t - i^t \le 2\tau|i - \hat{i}|$, we get

$$|V(x,\hat{i},t) - V(x,i,t)| \le \frac{1}{t}(\hat{i}^t - i^t) + \frac{1}{t}|i^0 - \hat{i}^0| \le \frac{2\tau+1}{t}|i-\hat{i}| \stackrel{t\to\infty}{\longrightarrow} 0.$$

Thus $V(x,i) = V(x,\hat{i})$. Using the same argument one extends this result to neighboring positive particles, and repeating it to all positive particles in the configuration. □

Corollary 2.5 *The invariance of velocities of negative particles can be proven along the same argument. Using these results we may drop the dependence on the index i in the definition of the average velocity.*

3 Dynamics of (Proto)clusters and Their Basins of Attraction (BA)

For a collection of consecutive particles of the same sign ordered with respect to their sign (i.e. according to their positions in the case of the positive sign, and in the opposite way in the case of the negative sign) we call *rear/leading* the first/last particle with respect to this order. By a *cluster* (of particles) we mean a segment $J = x[m', m]$, $m' < m$ of the configuration x consisting only of particles of the same sign (called the *body* of the cluster) and in which the leading particle is interacting with a particle of the opposite sign (called the *root* of the cluster) and the site next to the rear particle is vacant or is occupied by a particle of the same sign as the root. The sign of a cluster is identified with the sign of particles in its body. Depending on the type of the interaction between the leading particle and the root we say that the cluster has a *short* or *long* root. The right hand side of the diagram below shows positive clusters with short and long roots, while the left hand side demonstrates how these clusters may be created:

$$\cdots \underbrace{+\cdots+}_{\text{body}} \cdots \underbrace{\;-\;}_{\text{root}} \cdots \Longrightarrow \begin{cases} \cdots \underbrace{++}_{\text{body}}\ \underbrace{\;-\;}_{\text{short root}} \cdots \\ \cdots \underbrace{++}_{\text{body}} \cdot \underbrace{\;-\;}_{\text{long root}} \cdots \end{cases}$$

In distinction to other deterministic transport flow models (see e.g. [3–6]) in the model under consideration new clusters might be created even in the absence of clusters in the initial configuration.

In order to control events of this sort we introduce the notion of a *protocluster*[6] as a pair of consecutive particles of the same sign and a particle of the opposite sign satisfying the property that these three particles will form a true cluster and the moment when the leading of these two particles will start interacting with the rear one is the moment of the creation of a new cluster. In other words, they do not join an existing cluster. The minimal lattice segment containing this pair of particles is called the *body* of a protocluster and the particle of the opposite sign with which they will form the cluster—its *root*. The sign of a protocluster is again identified with the sign of particles in its body. Example of a positive protocluster and clusters which it may form are shown in the diagram above. Note that between the body and the root of a protocluster there might be some other particles of the same sign as the body,

[6] In the analysis of the free phase we will need to prove that the conditions of Lemma 3.3 imply that a flow of noninteracting at time $t \ge 0$ positive particles remain free under dynamics.

which even might form a cluster with the root but will stop interacting with the root particle before the leading particle of the protocluster will reach them.

Remark 3.1 A simple calculation shows that particles of the same sign as the root might be present between the body of the protocluster and its root only if they will make short interactions with the particles from the body while with the root they will make long interactions. Indeed, only in this case a new cluster can be created.

The boundaries of the body of a (proto)cluster might change with time when the particles are moving (in the case of a protocluster) or the leading particle leaves it while some particles join the cluster from the other side. At time $t \ge 0$ by J^t we denote the segment of the configuration corresponding to the body of the (proto)cluster originally located at $J := J^0$. If $t = 0$ we drop the time index to simplify the notation. The duration of time during which the cluster exists (i.e. the number of particles in its body is not smaller than two and the leading one is interacting with the same root particle) we call the *life-time* of the cluster J^t and denote by $LT(J)$. By the *life-time* of a protocluster we mean the duration of time until the cluster formed by particles from the body of the protocluster ceases to exist. If during its life-time the body of a (proto)cluster consists of only two particles (being present through its entire life-time) we call it *trivial* and *nontrivial* in the opposite case. Clearly the life-time of a trivial (proto)cluster is finite and depends only on the distance to the root. Therefore we shall be interested only in the case of nontrivial (proto)clusters. Observe that the situation described in Remark 3.1 corresponds to the trivial protocluster.

'Attracting' preceding particles, a (proto)cluster J plays a role similar to an attractor in dynamical systems theory. Therefore it is reasonable to study it in a similar way and to introduce the notion of its *basin of attraction* (notation $\mathrm{BA}(J)$), by which we mean the minimal segment of the configuration x containing all sites from where particles may eventually join the (proto)cluster during its life-time.

Due to the duality between positive and negative particles it is enough to consider only positive (proto)clusters. Let $J = x[m', m],\ m' < m$ be the body of a positive (proto)cluster having its root at site $m'' > m$. To take into account the dynamics we use the notation $J^t = x^t[m'(t), m(t)]$ and $m''(t) > m(t)$ for the corresponding objects at time $0 \le t \le \mathrm{LT}(J)$. We introduce two functionals depending on an integer parameter $k < m'$:

$$W(x[k,m]) := q(x_m) + \sum_{i=k}^{m} w(x_i), \qquad W'(x[k,m]) := q(x_m) + \sum_{i=k}^{m} w'(x_i), \tag{3.1}$$

where

$$w(z) := \begin{cases} \tau + 1 & \text{if } z > 0, \\ -\tau & \text{if } z < 0, \\ -1 & \text{otherwise}, \end{cases} \qquad w'(z) := \begin{cases} \tau & \text{if } z > 0, \\ -\tau & \text{if } z < 0, \\ -1 & \text{otherwise}, \end{cases}$$

and $q(z)$ is defined as the amount of time until the leading particle leaves the cluster, or $q(z) := \tau$ in the case of a true protocluster. Thus $q(z) = \tau - z + 1$ for the cluster with the short root, and $q(z) = \tau - z + 2$ for the cluster with the long root.

Denote by $j(t)$ the largest integer non-exceeding $m'(t)$ for which $W(x^t[j(t), m(t)]) = 0$ or $W(x^t[j(t) - 1, m(t)]) < 0$, and by $j'(t)$ the largest integer non-exceeding $m'(t)$ for which $W(x^t[j'(t), m(t)]) = 0$ or $W(x^t[j'(t) - 1, m(t)]) < 0$. Note that both $j(t)$ and $j'(t)$ might take an infinite value.

The following result allows to control the left boundary of a BA.

Theorem 3.1 $x[j'(t), m(t)] \subseteq \mathrm{BA}(J^t) \subseteq x[j(t), m(t)]$ *and* $j(t+1) \geq j(t)+1,\ j'(t+1) = j'(t)+1$.

Proof The proof of this theorem will be given through a series of lemmas, but before to do this consider an example demonstrating that the boundaries j, j' might differ a lot and the quantity $j' - j$ even might take an infinite value. Consider a spatially periodic configuration x with the main period containing three particles and three vacancies: $+\cdot\cdot+\cdot-$. These three particles represent a protocluster for which $j(0) = -\infty,\ j'(0) = i - 2$ where i stands for the position of the 1st positive particle.

Lemma 3.2 *The life-time of a cluster of positive particles is equal to the number of positive particles in its* BA *multiplied by* τ *minus the state of the leading particle plus one in the case of the short root or plus two in the case of the long root.*

Proof The term "the number of positive particles in its BA multiplied by τ" describes the amount of time before all particles in the BA will leave the cluster, assuming that the initial state of the leading particle is one. Indeed, during the life-time of a cluster the current leading particle is making continuously the short interaction with the root particle. After this the role of the leading particle is going to the next particle in the original BA. The term "the state of the leading particle plus one..." takes into account the actual state of the leading particle and the type of interactions with the root. □

Lemma 3.3 $\mathrm{BA}(J) \subseteq x[j, m]$ *and the life-time of the cluster* J *can be estimated from above as the number of positive particles in* $x[j, m]$ *multiplied by* $(\tau + 1)$.

Proof Consider the segment $I := x[j, m]$, set $n := m - j + 1$, and denote by N the number of positive particles in this segment (which cannot be smaller than 2 by the definition of the cluster) and by K the number of vacancies in it. Then the number of negative particles in this segment is equal to $n + 1 - N - K$. Assume contrary to our claim that there exists a particle ξ located initially to the left from the site j which joins the cluster $J := x[m', m]$ during its life-time. Since both vacancies and negative particles are moving to the left (opposite to the movement of positive particles) the particle ξ must interact with all these vacancies and negative particles before to join the cluster J. The amount of time necessary in order to do this is at least $t' := K + (n + 1 - N - K)\tau$. By the definition of the functional $W(x[j, m])$ and the parameter j we have

$$N(\tau + 1) - K - (n + 1 - N - K)\tau \leq 0.$$

Using this inequality we estimate the time t' from above as $N(\tau + 1)$, which is larger than the upper estimate $N\tau + 1$ of the amount time during which all N positive particles will join the cluster J and then leave it. Therefore the particle ξ cannot join the cluster J in time. We came to the contradiction. □

Lemma 3.4 *Let* J *be a nontrivial protocluster, then* $\mathrm{BA}(J^t) \subseteq x[j(t), m(t)]$.

Proof Denote by $\xi(t)$ the position at time $t \geq 0$ of a particle being originally at time $t = 0$ to the left from the site $j(0)$. Our aim is to show that $\xi(t) < j(t)$ for any moment of time $t \geq 0$ until the cluster is formed. Whence it is formed we can use the result of Lemma 3.3.

Observe that during the initial part of the life-time of a protocluster the leading particle moves at rate one exchanging positions with vacancies until at time t_1 it starts interacting with the root. After this whence another particle from the body of the protocluster reaches the leading one at time $t_2 > t_1$ the cluster is formed and one can apply Lemma 3.3 to get the result. Therefore we are concerned only with these two initial parts of the dynamics of a protocluster mentioned above.

During the time $0 \le t < t_1$ the leading particle is moving freely exchanging its position with vacancies. Therefore

$$W(x^{t+1}[j(t), m(t+1)]) = W(x^t[j(t), m(t)]) - 1,$$

because a new vacancy is taken into account while all already present particles and vacancies contribute the same values as before.

Observe that the site $j(t)$ cannot be occupied by a positive particle, otherwise

$$W(x^t[j(t), m(t)]) \ge \tau + 1$$

and hence

$$W(x^t[j(t) - 1, m(t)]) > 0$$

which contradicts to the definition of $j(t)$. Thus the boundary $j(t)$ moves at least at rate one. On the other hand, the particle ξ also cannot move faster than at rate one. Therefore $\xi(0) \le j(0)$ implies that $\xi(t) \le j(t)$ during the initial part of the life-time of a protocluster.

During the time $t_1 \le t < t_2$ the leading particle stays at the same site and thus there are no new vacancies. However, the state of the leading particle is increasing at rate one which plays exactly the same role and leads to the same result as in the previous case. □

To deal with the jammed phase we need to consider the lower bound for the BA. To this end we apply the functional W' instead of W to show that the corresponding boundary is located inside of the BA and that all particles from the corresponding segment must join the cluster during its life-time.

Lemma 3.5 $\mathrm{BA}(J^t) \supseteq x[j'(t), m(t)]$.

Proof The argument here is very similar to the one used in the proof of Lemma 3.4 with the only difference that we need to show now that for any particle $\xi(t)$ initially located in the segment $x[j(0), m(0)]$ we have $\xi(t) \ge j(t)$. In other words, that the boundary $j(t)$ cannot outran the particle $\xi(t)$.

The state of the current leading particle of the cluster is increasing by one until it will exchange positions with the root particle. Therefore the value of the functional W' is decreasing at rate one. Observe now that the site $j'(t)$ cannot be occupied by a positive particle (otherwise $W(x^t[j'(t), m(t)]) \ge \tau$) and hence $W'(x^t[j'(t) - 1, m(t)]) \ge 0$ which contradicts to the definition of $j'(t)$). Thus the boundary $j'(t)$ moves at least by rate one. On the other hand, the particle ξ also cannot move faster than at rate one. Therefore $j(t)$ being initially smaller than $\xi(t)$ may outran the latter only if at some time t' it makes a jump longer than one. This can happen only if there is i such that $j(t'-1) < i < m(t'-1)$ and $W'(x^t[i, m(t'-1)]) = 1$. In this case one might expect that on the next time step we have $W'(x^{t'}[i, m(t')]) = 0$. However this happens only if the segment $x^{t'}[i-1, i]$ at time $t'-1$ is occupied only by vacancies, which in turn means that $W'(x^{t'}[i-1, m(t'-1)]) = 0$ and hence $i - 1 = j(t')$. Thus long boundary jumps cannot take place. □

This finishes the proof of Theorem 3.1. □

The functional W' is everywhere positive in the case of an infinite BA and is used to calculate the second critical density corresponding to the beginning of the jammed phase:

$$\rho\tau - \tilde{\rho}\tau - (1-\rho-\tilde{\rho}) > 0 \quad\Longrightarrow\quad \rho(1+\tau) - \tilde{\rho}(\tau-1) > 1 \quad\Longrightarrow\quad \rho_c' = \frac{1+\tilde{\rho}(\tau-1)}{1+\tau}.$$

In the "intermediate phase" (see Sect. 6) and during the transient period of the free phase there are no infinite life-time clusters but instead there might be *recurrent* ones, which are appearing and disappearing again (in the intermediate phase) and finite life-time clusters even in the eventually free phase. To calculate the corresponding critical density ρ_c we need take into account that the duration of the long interaction is $\tau+1$ instead of τ. Therefore it can be calculated using the functional W in which the weight of a positive particle is $(\tau+1)$ but the weight of a negative particle is $-\tau$. The condition $W(x[m-n,m]) \le 0$ for some arbitrary large n gives

$$\begin{aligned} \rho(\tau+1) - \tilde{\rho}\tau - (1-\rho-\tilde{\rho}) > 0 \quad&\Longrightarrow\quad \rho(2+\tau) - \tilde{\rho}(\tau-1) > 1 \\ &\Longrightarrow\quad \rho_c = \frac{1+\tilde{\rho}(\tau-1)}{2+\tau}. \end{aligned}$$

4 Duality Relations and Calculation of Average Velocities

In this section we discuss connections between dynamics of particles of different types.

A configuration $(-x) \in X$ is said to be *dual* to x. This is indeed the case since the state of each particle is inverted from positive to negative and vice versa. It is straightforward to check that the dynamics of configurations x^t and $-x^t$ are exactly the same except for the direction of the particles movement. This type of relations we call *duality* and they allow to simplify significantly the analysis of such systems. In particular we shall mainly discuss properties of positive particles having in mind that the corresponding properties of negative particles can be obtained by duality relations.

For a configuration $x \in X$ with densities of positive and negative particles $\rho(x)$ and $\tilde{\rho}(x)$ respectively denote by $V(\rho,\tilde{\rho})$ the average velocity (if it exists) of positive particles and by $\tilde{V}(\rho,\tilde{\rho})$ the modulus of the average velocity of negative particles. By the duality between positive and negative particles we have $V(\rho,\tilde{\rho}) = \tilde{V}(\tilde{\rho},\rho)$.

It turns out that the assumption about the existence of average velocities together with some information about the structure of configurations allows to calculate the average velocities in terms of particle densities.

Lemma 4.1 *Assume that in a configuration x the densities and average velocities of both types of particles are well defined and that positive particles move freely* (*i.e. except from the interactions with negative particles*). *Then*

$$V(\rho,\tilde{\rho}) = 1 - (\tau-1)\tilde{\rho}(V(\rho,\tilde{\rho}) + \tilde{V}(\rho,\tilde{\rho})). \tag{4.1}$$

Proof During the time $t > 0$ a positive particle will meet $(V(\rho,\tilde{\rho}) + \tilde{V}(\rho,\tilde{\rho})) \cdot t\tilde{\rho} + o(t)$ negative ones. The notation $o(t)$ stands for a term which grows slower than t, i.e.

$o(t)/t \overset{t\to\infty}{\longrightarrow} 0$. Thus taking into account that according to the assumption positive particles never wait in clusters (which otherwise will give an additional contribution) we have:

$$V(\rho,\tilde{\rho})\cdot t = t-(\tau-1)(V(\rho,\tilde{\rho})+\tilde{V}(\rho,\tilde{\rho}))\cdot t\tilde{\rho}+o(t).$$

Dividing both sides of this relation by t and passing to the limit as $t\to\infty$ we get the result. □

Lemma 4.2 *Assume additionally to the assumptions of Lemma 4.1 that negative particles are free also. Then*

$$V(\rho,\tilde{\rho})=\frac{1+(\tau-1)(\rho-\tilde{\rho})}{1+(\tau-1)(\rho+\tilde{\rho})}. \tag{4.2}$$

Proof By Lemma 4.1 we have

$$V(\rho,\tilde{\rho})=1-(\tau-1)\tilde{\rho}(V(\rho,\tilde{\rho})+\tilde{V}(\rho,\tilde{\rho})),$$
$$\tilde{V}(\rho,\tilde{\rho})=1-(\tau-1)\rho(V(\rho,\tilde{\rho})+\tilde{V}(\rho,\tilde{\rho})).$$

Denote $v=V(\rho,\tilde{\rho})+\tilde{V}(\rho,\tilde{\rho})$. Then adding the above relations we get $v=2-(\tau-1)\times(\rho+\tilde{\rho})v$, which yields

$$v=\frac{2}{1+(\tau-1)(\rho+\tilde{\rho})}$$

and hence

$$V(\rho,\tilde{\rho})=\frac{1+(\tau-1)(\rho-\tilde{\rho})}{1+(\tau-1)(\rho+\tilde{\rho})}.$$

In particular,

$$V(\rho,\rho)=\frac{1}{1+2(\tau-1)\rho}.$$ □

Lemma 4.3 *Assume now that all negative particles are free but positive particles form infinite life-time clusters. Then* $V(\rho)=\frac{1}{\tau}(\frac{1}{\rho}-1)$.

Proof According to our assumptions after at most $\tau+1$ time steps the leading particles in the infinite life-time cluster will take part only in short interactions with the root of the cluster. Consider now the flux of positive particles through the root of the cluster. According to the argument above after at most $\tau+1$ iterations the flux become equal to $1/\tau$ (i.e. after each τ moments of time a leading particle from the cluster body exchanges positions with the cluster root). Thus $1/\tau=\rho(V(\rho,\tilde{\rho})+\tilde{V}(\rho,\tilde{\rho}))$ and hence $V(\rho)=\frac{1}{\tau}(\frac{1}{\rho}-1)$. □

Observe that the difference between short and long interactions does not matter both in the free and jammed phases because in the former it does not change the timing, while in the latter in the steady state it cannot happen if the length of the jam is greater than 1 (there are no gaps between particles in the cluster body).

The derivation of the average velocity in the free phase is based on the assumption that $\tilde{V}(\rho,\tilde{\rho})$ describes the flow of noninteracting particles. If the density of negative particles is becoming large enough then they start interact between themselves. Assume that this density is so large that the corresponding velocity belongs to the jammed region and hence

$$V(\rho,\tilde{\rho}) = 1-(\tau-1)(V(\rho,\tilde{\rho})+\tilde{V}(\rho,\tilde{\rho}))\tilde{\rho},$$
$$\tilde{V}(\rho,\tilde{\rho}) = \frac{1}{\tau}\left(\frac{1}{\tilde{\rho}}-1\right).$$

Then $V(\rho,\tilde{\rho}) = 1/\tau$ which does not depend on the particle density and corresponds to the case when all the time a positive particle interacts with negative ones, i.e. the negative particles form an infinite life-time cluster. Clearly this means that if a configuration belongs to $\mathrm{Free}_+^\infty \cap \mathrm{Jam}_-^\infty$ then the average velocity of positive particles is equal to $1/\tau$.

5 Proof of Theorems 1.2 and 1.1

Proof These results will be proven more or less simultaneously through a series of technical lemmas.

Lemma 5.1 *Let a configuration x satisfies the condition*

$$(\tau+2)\rho(x)-(\tau-1)\tilde{\rho}(x) < 1. \tag{5.1}$$

Then only finite life-time (proto)clusters may be present in the configuration x. Moreover, there exists a partition of the integer lattice into nonoverlapping BA*s of these finite life-time (proto)clusters and their complements.*

Proof Assume on the contrary that there is a (proto)cluster $J = x[m', m]$ with an infinite BA. By Theorem 3.1 and the definition of the functional W for any integer $n > 1$ we have $W(x[m-n, m]) > 0$. Rewriting this inequality in terms of particle densities we get

$$\begin{aligned} q(x_m) + (\tau+1)n\rho(x,[m-n,m-1]) - \tau n\tilde{\rho}(x,[m-n,m-1]) \\ - n(1-\rho(x,[m-n,m-1]) - \tilde{\rho}(x,[m-n,m-1])) > 0. \end{aligned}$$

Dividing by n and passing to the limit as $n\to\infty$ we obtain

$$(\tau+2)\rho(x)-(\tau-1)\tilde{\rho}(x) \ge 1,$$

which contradicts to the assumption (5.1).

To prove the second claim we need to demonstrate that the finite BAs cannot be infinitely nested. The latter means that there exists a sequence of (proto)clusters $\{J_n\}_n$ and an integer N such that for any $n > N$ we have $\mathrm{BA}(J_n) \supset \mathrm{BA}(J_N)$. If this would be the case a positive particle located initially to the right of J_N will never become free despite the amount of time it spends in each of the clusters is finite.

Assume that such an infinitely nested sequence of (proto)clusters $\{J_n\}_n$ does exist. Then for some $m \in \mathbb{Z}$ and any integer $n > 1$ by Theorem 3.1 we have $W(x[m, m+n]) > 0$. Rewriting this in terms of particle densities we get

$$\begin{aligned} q(x_{m+n}) + (\tau+1)n\rho(x,[m,m+n-1]) - \tau n\tilde{\rho}(x,[m,m+n-1]) \\ - n(1-\rho(x,[m,m+n-1]) - \tilde{\rho}(x,[m,m+n-1])) > 0. \end{aligned}$$

Exactly as in the previous case we divide this inequality by n and after passing to the limit as $n \to \infty$ we come to the contradiction to the assumption (5.1). □

Remark The importance of the 2nd claim of Lemma 5.1 is that it might be possible that all clusters have only finite life-times but a particle is joining them one after another and is never becoming free.

Lemma 5.2 *Let a configuration x satisfy the conditions of Lemma 5.1. Then any positive particle after some initial transient period will stop interacting with other particles of the same sign (i.e. will become free).*

Proof By the previous Lemma there exists a partition of the integer lattice into nonoverlapping BAs of finite life-time (proto)clusters and their complements. Choose one of these BAs and consider a particle ξ located originally immediately before this BA. By the definition of the BA the particle ξ does not belong to any BA of (proto)clusters located ahead of it and hence it will never join any cluster of positive particles. □

Results of Lemmas 5.1, 5.2 prove claims (a, b) of Theorem 1.1.

Lemma 5.3 *Let a configuration x satisfy the condition*

$$(\tau + 1)\rho(x) - (\tau - 1)\tilde{\rho}(x) > 1. \tag{5.2}$$

Then for each $N \in \mathbb{Z}$ there is an infinite life-time (proto)cluster in x located to the right from the site N.

Proof Fix some $N \in \mathbb{Z}$ and assume on the contrary that any positive particle in x located at a site $i > N$ is either free or belongs to a finite life-time (proto)cluster.

Thus there exists a sequence of finite BAs of (proto)clusters J_k such that $\mathrm{BA}(J_k) \supseteq [j'_k, m_k]$ and $j'_k < m_k < j'_{k+1} < m_{k+1} \xrightarrow{k\to\infty} \infty$. By the definition of j'_k for any $\ell > 1$ we have

$$\sum_{k=1}^{\ell} W'(x[j'_k, m_k]) < 0.$$

Between the segments $[j'_k, m_k]$ there might be some additional positive particles not belonging to any BA and thus each of them satisfies the condition that either the number of vacancies immediately ahead of it exceeds τ or it is preceded by a negative particle. In both cases (as well if there are additional negative particles) the value of the functional W' on the complete segment $[j'_1, m_\ell]$ instead of the union of segments $[j'_k, m_k]$ is again negative.

Denoting $m := j'_1,\ n_\ell := m_\ell - j'_1$ and rewriting the inequality $W'(x[m, m + n_\ell - 1]) \le 0$ in terms of particle densities we get

$$\begin{aligned} q(x_{m+n_\ell}) &+ (\tau + 1)n\rho(x, [m, m + n_\ell - 1]) - \tau n\tilde{\rho}(x, [m, m + n_\ell - 1]) \\ &- n(1 - \rho(x, [m, m + n_\ell - 1]) - \tilde{\rho}(x, [m, m + n_\ell - 1])) < 0. \end{aligned}$$

Dividing this inequality by n and passing to the limit as $\ell \to \infty$ we obtain

$$(\tau + 1)\rho(x) - (\tau - 1)\tilde{\rho}(x) \le 1.$$

Thus we came to the contradiction to the assumption (5.2). □

Claims (c, d) of Theorem 1.1 follow from this result.

In the hysteresis phase the protoclusters *might* be present, which makes the main difference to the free phase. Moreover, it might be possible that despite the protoclusters have only finite life-times their BAs are infinitely nested which prevents positive particles to become free eventually.

Now we are ready to prove Theorem 1.2.

We start with the case $(\rho(x), \tilde{\rho}(x)) \in A$. Then the particle densities satisfy the system of inequalities

$$(\tau+2)\rho < 1 + (\tau-1)\tilde{\rho},$$
$$(\tau+2)\tilde{\rho} < 1 + (\tau-1)\rho.$$

By Lemma 5.2 and the first of these inequalities all positive particles will eventually become free. On the other hand, applying the duality relation together with the second inequality, by Lemma 5.2 we get the same property for negative particles. Therefore exact relations for the average velocities $(V(x), \tilde{V}(x))$ follow from Lemma 4.2.

Assume now that $(\rho(x), \tilde{\rho}(x)) \in C$. Then

$$(\tau+1)\rho > 1 + (\tau-1)\tilde{\rho},$$
$$(\tau+2)\tilde{\rho} < 1 + (\tau-1)\rho.$$

The first of these inequalities implies by Lemma 5.3 that in the configuration x there are infinitely many infinite life-time (proto)clusters of positive particles, while from the second inequality it follows that all negative particles will eventually become free. Thus some negative particle will become a short root of an infinite life-time cluster and from that moment of time will move exactly by one site in τ time steps (exchanging positions with positive particles from the body of the cluster). Now since the average velocity is well defined for a single particle, by Lemma 2.4 the same result holds for all particles of the same sign and thus $\tilde{V}(x) = 1/\tau$. Once we get this we obtain the relation for the average velocity for positive particles by the duality relation.

The case $(\rho(x), \tilde{\rho}(x)) \in B$ is considered similarly to the previous one, except for in this case positive particles get the average velocity $1/\tau$.

Theorems 1.2 and 1.1 are proven. □

6 Hysteresis Region

Our previous results are concerned with the behavior of configurations in "pure" free and jammed phases. Now we shall show that in the region H located between these phases (see Fig. 1 (right)) the coexistence of free and jammed configurations takes place. In other words in this region there might be both freely moving and jammed configurations with the same particle densities. Due to the duality between positive and negative particles in what follows we consider only the case when $\rho \geq \tilde{\rho}$.

Lemma 6.1 *Configurations from the set* Free_+^∞ *are dense in* $H_+ := \{(\rho, \tilde{\rho}) \in H : (\tau+1)\rho \leq 1 + (\tau-1)\tilde{\rho}\}$.

Proof Consider a family of spatially periodic configurations parameterized by nonnegative integers $n, m, k_1, \dots, k_n$ (only one spatial period is shown):

$$\overbrace{+\underbrace{\cdots}_{\tau+2k_1}+\underbrace{\cdots}_{\tau+2k_2}\dots+\underbrace{\cdots}_{\tau+2k_n}}^{(\tau+1)n+2N}\overbrace{\underbrace{+-}\;\underbrace{+-}\dots\underbrace{+-}}^{2m} \tag{6.1}$$

Setting $N := \sum_{i=1}^{n} k_i$ the length of the period can be written as $(\tau+1)n + 2(m+N)$. The value $\tau + 2k_i$ is equal to the duration of the short interaction plus an arbitrary even number. This guarantees that only short interactions may take place under dynamics. On the other hand, the number of vacancies between consecutive positive particles is not smaller than τ, which implies that for any k_i a positive particle cannot catch up with another positive particle (since only short interactions may take place). Therefore $x \in \mathrm{Free}_+$.

Let a configuration x belongs to the family (6.1). Then

$$\rho(x) = \frac{n+m}{(\tau+1)n + 2(m+N)} =: r(n,m,N),$$
$$\tilde{\rho}(x) = \frac{m}{(\tau+1)n + 2(m+N)} =: \tilde{r}(n,m,N).$$

Observe now that for any $u, v, z \ge 0$ such that $u + v > 0$ the identities

$$r(u,v,z) \equiv r\left(\frac{u}{u+v}, \frac{v}{u+v}, \frac{z}{u+v}\right), \qquad r(u,v,z) \equiv r\left(\frac{u}{u+v}, \frac{v}{u+v}, \frac{z}{u+v}\right)$$

hold true and that

$$1 + (\tau-1)\tilde{r}(u,v,z) - (\tau+1)r(u,v,z) = \frac{2z}{(\tau+1)u + 2(v+z)} \ge 0.$$

Therefore the "right" boundary of the region H (corresponding to the case $z = 0$) contains configurations from Free_+. Moreover, the points on the "right" boundary for which the above representation takes place are dense. Additionally we see that these functions decay with respect to each their argument. Therefore for a given pair or real numbers $u, v \ge 0,\ u + v > 0$ the pair of functions $(r(u,v,z),\ \tilde{r}(u,v,z))$, considered as functions of the third real argument z, parameterizes a curve $R_{u,v}(z) := (r(u,v,z),\ \tilde{r}(u,v,z))$ which starts (when $z = 0$) at some point on the "right" boundary of H and goes through H until it reaches the "left" boundary.

For $d \ge 1$ denote $\varrho_d((u_1,\dots,u_d),(u'_1,\dots,u'_d)) := \sum_i |u_i - u'_i|$. Then

$$\varrho_2(R_{u,v}(z), R_{u',v'}(z')) \le \frac{2\tau\varrho_3((u,v,z),(u',v',z'))}{(u+v+z)(u'+v'+z')} \tag{6.2}$$

for any pair of triples $(u,v,z),\ (u',v',z')$. Therefore the curves $R_{u,v}$ with $u, v \in \mathbb{R}_+ \cup \{0\}$ fill in the entire region H_+.

It remains to show that for any real $(\rho, \tilde{\rho}) \in H_+$ and any $\varepsilon > 0$ there exists a triple of nonnegative integers (n, m, N) such that

$$\varrho_2((\rho,\tilde{\rho}), R_{n,m}(N)) < \varepsilon.$$

To do this set $n' := \frac{K\rho}{\rho+\tilde{\rho}},\ m' := \frac{K\tilde{\rho}}{\rho+\tilde{\rho}}$. Then for any $K > 0$ the curve $R_{n',m'}$ goes through the point $(\rho, \tilde{\rho})$ at some real parameter value N'. Since points on the "right" boundary of

H represented by integer values of (n, m, N) are dense we may choose two sequences of nonnegative integers $\{n_k\}, \{m_k\} \stackrel{k\to\infty}{\longrightarrow} \infty$ such that

$$\varrho_2(R_{n',m'}(0), R_{n_k,m_k}(0)) \stackrel{k\to\infty}{\longrightarrow} 0.$$

Therefore setting

$$u' := \frac{\rho}{\rho+\tilde\rho}, \qquad v' := \frac{K\tilde\rho}{\rho+\tilde\rho}, \qquad z' := \frac{N'}{n'+m'};$$
$$u_k := \frac{n_k}{n_k+m_k}, \qquad v_k := \frac{m_k}{n_k+m_k}, \qquad z_k := \frac{N}{n_k+m_k};$$

we get

$$\varrho_2((u_k, v_k), (u', v')) \stackrel{k\to\infty}{\longrightarrow} 0.$$

Using (6.2) we obtain

$$\begin{aligned}\varrho_2(R_{u_k,v_k}(z_k), R_{u',v'}(z')) &\le \frac{2\tau\varrho_3((u_k, v_k, z_k), (u', v', z'))}{(1+z_k)(1+z')}\\ &\le 2\tau(\varrho_2((u_k, v_k), (u', v')) + |z_k - z'|).\end{aligned}$$

Therefore the curves R_{n_k,m_k} come arbitrary close to the point $(\rho, \tilde\rho)$ as $k\to\infty$ at some integer parameter values N, for which

$$|z_k - z'| \stackrel{k\to\infty}{\longrightarrow} 0$$

and this finishes the proof. □

To study the "left" boundary of H consider another family of spatially periodic configurations:

$$\overbrace{+\underbrace{\cdots}_{\tau+1+2k_1}+\underbrace{\cdots}_{\tau+1+2k_2}\cdots+\underbrace{\cdots}_{\tau+1+2k_n}}^{(\tau+2)n+2N}\overbrace{\underbrace{+\cdot-}\,\underbrace{+\cdot-}\cdots\underbrace{+\cdot-}}^{3m} \tag{6.3}$$

The notation here is the same as for the family (6.1) and the length of the period is equal to $(\tau+2)n + 2(m+N)$. A direct check shows that again in this family both positive and negative particles are free. The difference is the absence of long interactions under dynamics for configurations from (6.1) and their inevitable presence for configurations from (6.3) for an arbitrary large time. If a configuration x belongs to the family (6.3) then

$$\rho(x) = \frac{n+m}{(\tau+2)n+2(m+N)}, \qquad \tilde\rho(x) = \frac{m}{(\tau+2)n+2(m+N)}.$$

Setting again $N = 0$ we obtain the "left" boundary of H, namely

$$(\tau+2)\rho(x) \equiv 1 + (\tau-1)\tilde\rho(x).$$

On the other hand, for the first family

$$1 \ge (\rho-\tilde\rho)(\tau+1) + 2\tilde\rho \quad\Longrightarrow\quad \rho \le \frac{1+\tilde\rho(\tau-1)}{\tau+1}$$

while for the second family

$$1 \geq (\rho - \tilde{\rho})(\tau + 2) + 3\tilde{\rho} \quad \Longrightarrow \quad \rho \leq \frac{1 + \tilde{\rho}(\tau - 1)}{\tau + 2}.$$

Comparing these inequalities with the critical densities ρ_c' and ρ_c we see that the family (6.3) belongs to the free phase while the family (6.1) does not belong there.

It remains to show that in this region there are jammed (for all times) configurations as well. According to Lemma 5.1 the inequality (5.1) (defining the "right" boundary of H_+) guarantees the absence of infinite life-time (proto)clusters in H. This, together with the denseness of densities corresponding to free configurations in H (see Lemma 6.1) explains the subtlety of this problem: one needs to find configurations having only (proto)clusters of arbitrary large but finite life-time. To understand how this can happen consider the dynamics of a spatially periodic configuration $x \in X$ with $\tau = 2$:

$$\begin{array}{c}
10001\tilde{1}1\tilde{1} \\
01002\tilde{2}2\tilde{2} \\
0010\tilde{1}1\tilde{1}1 \\
1020\tilde{2}2\tilde{2}0 \\
0130\tilde{3}\tilde{1}10 \\
01\tilde{1}01\tilde{1}01 \\
12\tilde{2}02\tilde{2}00 \\
1\tilde{1}10\tilde{1}100 \\
2\tilde{2}20\tilde{2}010 \\
\tilde{1}130\tilde{3}001 \\
\tilde{2}1\tilde{1}01002 \\
12\tilde{2}0010\tilde{1}
\end{array}$$

Using the notation introduced in the proof of Lemma 6.1 we have $n = 3, m = 2, N = 3$. Thus $r(n, m, N) = 3/8$, $\tilde{r}(n, m, N) = 1/4$ and hence

$$(\tau + 1)r(n, m, N) = 9/8 < 1 + (\tau - 1)\tilde{r}(n, m, N) = 5/4 < (\tau + 2)r(n, m, N) = 3/2,$$

i.e. x belongs to the region H. During the dynamics each positive particle in turn[7] makes short and long interactions with negative ones. Finite (proto)clusters are present for all time. On the other hand only trivial life-time clusters are present (at moments of time when one of positive particles catches up with another), hence the configuration in this example does not belong to the jammed phase. Nevertheless at any moment of time there are infinitely nested BAs of trivial protoclusters.

Our numerical simulations, which were carried out for a broad region of spatial periods L (up to $L = 200$) indicate that for each spatial period L for all pairs of positive integers n, m for which $n + m < L$ and $(r(n, m, L - n - m), \tilde{r}(n, m, L - n - m)) \in H$ there were configurations with only trivial (proto)clusters for arbitrary large time. This allows to formulate the following

Hypothesis *For each pair of densities belonging to the region H there exists an admissible configuration having positive and negative particles with these densities for which only trivial (proto)clusters are present for arbitrary large time.*

[7] The sequence of types of interactions for a single particle is periodic but may be arbitrary.

For spatially periodic configurations the existence of average velocities follows from the obvious time-periodicity of the dynamics of particles. However in the general case this is not proven. Moreover, without the assumption about the existence of particle densities one can construct configurations for which average velocities do not exist.

7 Dynamics of an Active Tracer

To model the motion of an active tracer in the traffic flow we add to the flow a single particle (representing the tracer). If the tracer moves in the same direction as the flow we think about this particle as having the same sign as particles of the flow, and having the opposite sign when the tracer moves against the flow. In distinction to the passive tracer studied in [5] the active one changes the behavior of the flow due to interactions with other particles. Clearly a single positive particle added to the flow of positive particles do not change its asymptotic behavior. On the other hand the case when the tracer moves against the flow is equivalent to the model discussed in this paper with zero density of positive particles (tracer) and a certain density of negative particles (the flow). According to Theorem 1.2 we get the following average velocity of the active tracer as a function of the density of the flow ρ:

$$V_+(\rho) = \begin{cases} 1 & \text{if } 0 \le \rho \le 1/2, \\ 1/\rho & \text{otherwise,} \end{cases}$$

$$V_-(\rho) = \begin{cases} \frac{1-(\tau-1)\rho}{1+(\tau-1)\rho} & \text{if } 0 \le \rho < \frac{1}{\tau+2}, \\ \frac{1}{\tau} & \text{if } \frac{1}{\tau+1} < \rho \le 1. \end{cases}$$

Here the average velocity $V_+(\rho)$ corresponds to the case when the tracer moves in the same direction as the flow, while $V_-(\rho)$ describes the case when the tracer moves in the opposite direction. Figure 2 shows that there is a region of flow densities when it is more advantageous to move against the flow than along it. The explanation is that in high density region the time necessary to exchange positions with a particle moving in the opposite direction is becoming smaller than the time a particle from the flow spends in inevitably being present large clusters.

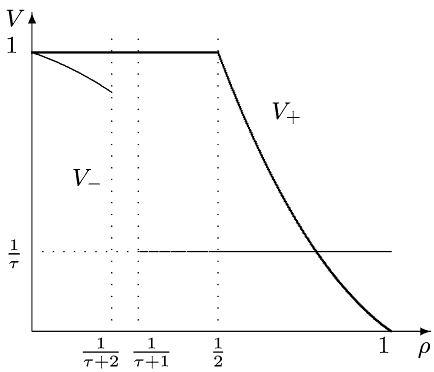

Fig. 2 Dependence of the average velocity of the active tracer $V_\pm$ on the particle density ρ. V_+—along the flow (*thick line*), V_-—against it (*thin line*)

8 Configurations Having No Particle Densities

So far we were discussing statistical properties of configurations for which both positive and negative particle densities are well defined. In general one or both of these densities might be not well defined and one needs to deal with lower and upper densities. Recall that Lemma 2.1 claims their invariance with respect to dynamics. The following result extends the description of the Phase diagram for this more general case.

Theorem 8.1 *Let* $x \in X$. *If*

(a) $(\tau+2)\rho^+(x) < 1 + (\tau-1)\tilde{\rho}^-(x)$ *then* $x \in \mathrm{Free}_+^\infty$.
(b) $(\tau+2)\tilde{\rho}^+(x) < 1 + (\tau-1)\rho^-(x)$ *then* $x \in \mathrm{Free}_-^\infty$.
(c) $(\tau+1)\rho^-(x) > 1 + (\tau-1)\tilde{\rho}^+(x)$ *then* $x \in \mathrm{Jam}_+^\infty$.
(d) $(\tau+1)\tilde{\rho}^-(x) > 1 + (\tau-1)\rho^+(x)$ *then* $x \in \mathrm{Jam}_-^\infty$.

The proof is similar to the proof of Theorem 1.1 and therefore we skip it.

In distinction to the case when the densities are well defined, here we cannot prove the existence of average velocities. Instead one considers upper and lower particle velocities for which it is possible to get a representation of the same sort as in Theorem 1.2.

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